The terms $(1/R^m)F_m w$ and $(1/R^m)\,F_m{}^*(\partial w/\partial x)$, $m = 0, \ldots, 3$, in (6a) are called "correction terms"; the following terms are referred to as "remainder." It is obvious that the remainder tends to zero as soon as one of the parameters M,R,ω, and $|\nu|$ tends to infinity.

An inspection of Eqs. (6) leads to the following conclusions:

$$\Delta p - \left\{\Delta p^{**} - \frac{a_0{}^2\rho_0}{2R}\,w\right\} \to 0 \qquad (7)$$

for $|M| \to \infty$

$$\Delta p - \left[\Delta p^{**} - \frac{1}{2}\,a_0\rho_0\left\{\frac{1}{M_1M_2} + \left|\binom{-\frac{1}{2}}{2}\right| \frac{(\nu a_0/\omega)(M_2{}^3 - M_1{}^3)}{M_1{}^3M_2{}^3} + \left|\binom{-\frac{1}{2}}{3}\right| \frac{(\nu a_0/\omega)(M_2{}^5 - M_1{}^5)}{M_1{}^5M_2{}^5} + \ldots\right\}\frac{\partial w}{\partial t} + \frac{1}{2}\,a_0{}^2\rho_0 M\left\{\frac{1}{M_1M_2} + \left|\binom{-\frac{1}{2}}{2}\right| \frac{M_2{}^3 + M_1{}^3}{MM_1{}^3M_2{}^3} + \left|\binom{-\frac{1}{2}}{3}\right| \frac{M_2{}^5 + M_1{}^5}{MM_1{}^5M_2{}^5} + \ldots\right\}\frac{\partial w}{\partial x}\right] \to 0 \qquad (8)$$

for $R \to \infty$

The latter result is obtained after F_0 and $F_0{}^*$ have been expanded in power series of $1/M_1$ and $1/M_2$ and after some regrouping. Obviously, the terms $M_2{}^\mu - M_1{}^\mu$, $\mu = 3, 5, \ldots$, contain the factor $\omega/\nu a_0$. The terms $M_2{}^\mu + M_1{}^\mu$, $\mu = 3,5, \ldots$, contain the factor M. The relations (7) and (8) lead to

$$\Delta p - \Delta p^{**} \to 0 \qquad (9)$$

for $|M| \to \infty$; $R \to \infty$

Furthermore, one obtains

$$\Delta p - \{\Delta p^{**} - (a_0{}^2\rho_0/2R)w\} \to 0 \qquad (10)$$

for $\omega \to \infty$

Hence

$$\Delta p - \Delta p^{**} \to 0 \qquad (11)$$

for $R \to \infty$; $\omega \to \infty$

In order to investigate the limiting process $|\nu| \to \infty$, M is restricted by $|M| > 1$. Thanks to this inequality, the inequalities (5) are satisfied for sufficiently large values of $|\nu|$. Then one learns that

$$\Delta p - \left[\Delta p^{**} + a_0\rho_0\left(\frac{|M|}{(M^2-1)^{1/2}}\,\frac{M^2-2}{M^2-1} - 1\right)\frac{\partial w}{\partial t} + a_0{}^2\rho_0\left(\frac{M|M|}{(M^2-1)^{1/2}} - M\right)\frac{\partial w}{\partial x} - \frac{a_0{}^2\rho_0}{2R}\,\frac{M^2}{M^2-1}\,w\right] \to 0$$

for $|\nu| \to \infty$ (12)

From (12) one arrives at

$$\Delta p - \{\Delta p^{**} - (a_0{}^2\rho_0/2R)w\} \to 0 \qquad (13)$$

for $|M| \to \infty$, $|\nu| \to \infty$

$$\Delta p - \Delta p^{**} \to 0 \qquad (14)$$

for $|M| \to \infty$, $|\nu| \to \infty$, $R \to \infty$

These investigations demonstrate that, for the case $|M_1| > 1$; $|M_2| > 1$, the linear piston theory expression Δp^{**} can be considered as a first-order approximation of the aerodynamic pressure Δp. Nevertheless, the replacement of Δp^{**} by the approximation

$$\Delta p^{***} = \Delta p^{**} - \frac{a_0{}^2\rho_0}{2R}\,w = a_0\rho_0\left\{a_0 M\,\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} - \frac{a_0}{2R}\,w\right\} \qquad (15)$$

is suggested as a first-order step of improvement for the application to cylindrical shells. Improved approximations can be obtained from Eqs. (6). For large values of n, especially, the correction terms of the order $(1/R)^2$ and $(1/R)^3$ in Eqs. (6) can become quite significant. For flutter investigations, it often is necessary to consider values of n up to the order of 20.

Further investigations along this line (see Ref. 6) disclose that, in cases 1) $|M_1| < 1$, $|M_2| < 1$; 2) $|M_1| < 1$, $|M_2| > 1$; and 3) $|M_1| > 1$, $|M_2| < 1$, the linear piston theory expression Δp^{**} no longer can be considered as a first-order approximation for the aerodynamic pressure Δp.

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Transient Temperature of a Porous-Cooled Wall

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Nomenclature

C = specific heat, Btu/lb-°F
G = mass flow rate, lb/hr-ft²
k = thermal conductivity, Btu/hr-ft-°F
L = thickness, ft
q = heat flux, Btu/hr-ft²
t = temperature, °F
x = normal coordinate through wall, ft
θ = time, hr
ρ = density, lb/ft³

Subscripts

c = coolant
w = porous solid

THIS note presents parametric curves, based on an exact solution, for the prediction of the transient temperature response of a one-dimensional transpiration-cooled porous wall that is exposed instantaneously to a constant heat flux on the coolant exit surface. For short-time applications,

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the transient temperature response may be of considerable importance for the calculation of the coolant flow rate and the thermal stress distribution of the porous material. Analysis of a transient period based on a steady-state temperature solution may result in the requirement for an excessive amount of coolant to maintain a given wall temperature and an underestimate of the thermal stress based on a lower thermal gradient.

The constant heat flux condition chosen herein may be experienced in a heating facility such as an arc jet. It may be noted also that the constant heat flux condition results in a solution containing the minimum number of dimensionless groups, thereby simplifying graphical presentation and preliminary parametric study.

The problem considered in this note first was presented in Ref. 1 without derivation. The temperature of the coolant exit face was plotted over the range of Fourier numbers from 0.2 to 2.0 for various coolant flow rate parameters. Shortly afterward, the partial differential equation of Ref. 1, Eq. (1), was solved for a uniform-temperature convective condition.[2] The temperatures of the coolant exit and entrance faces were calculated over the range of Fourier numbers from 0.01 to 50 for various Biot and coolant flow parameters. In a later publication,[3] the problem of Ref. 1 and this note was solved by finite-difference approximation without reference to Ref. 1. The temperatures obtained in Ref. 3 are high when compared to the exact solution.

This note extends the work of Ref. 1 by presenting temperature curves of the coolant exit, entrance, and center face over a larger range of Fourier and coolant flow parameters. The nomenclature of Ref. 2 is followed herein to facilitate comparison.

Analysis

Initially, an infinite wall of finite thickness and uniform porosity is in thermal equilibrium with the coolant that flows through it, both being at the coolant reservoir temperature t_c. The flow of coolant G_c is steady for all times, entering the wall at $x = 0$ and exiting at $x = L$. At zero time the surface $x = L$, the exit face, is exposed to a constant heat flux q. The properties of the wall and coolant are taken as constant. In addition, the usual and reasonably justifiable assumptions that the solid and coolant temperatures are equal throughout the wall[4] and that the thermal conductivity of the coolant is small in comparison with that of the wall result in the following partial differential equation:

$$k_w(\partial^2 t/\partial x^2) - G_c C_c(\partial t/\partial x) = \rho_w C_w(\partial t/\partial \theta) \qquad (1)$$

with boundary and initial conditions

$$k_w[\partial t(0,\theta)/\partial x] = G_c C_c[t(0,\theta) - t_c] \qquad (2)$$

$$k_w[\partial t(L,\theta)/\partial x] = q \qquad (3)$$

$$t(x,0) = t_c \qquad (4)$$

These relations are nondimensionalized by introducing the Fourier number, mass flow rate parameter, temperature parameter, and distance parameter, respectively.

$$Fo = k_w\theta/\rho_w C_w L^2 \qquad g = G_c C_c L/k_w$$

$$u = (t - t_c)/(qL/k_w) \qquad \xi = x/L$$

The solution to the foregoing equations, obtained by the method of separation of variables, is†

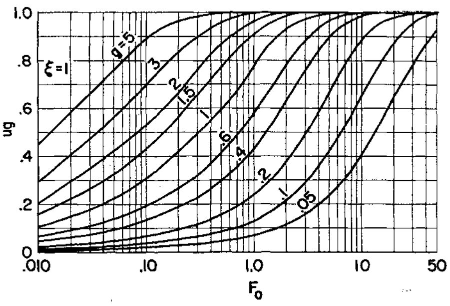

Fig. 1 Temperature response of exit face

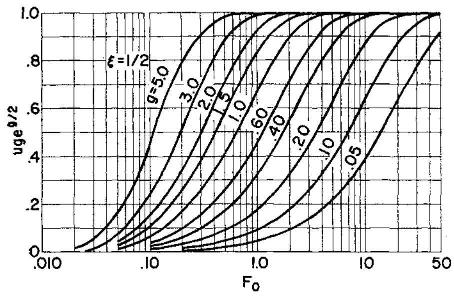

Fig. 2 Temperature response of center plane

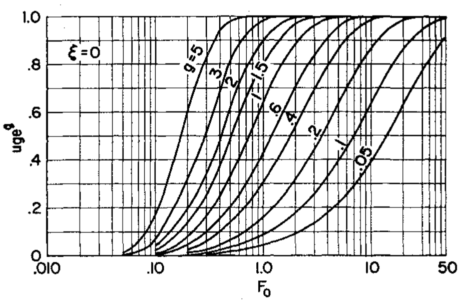

Fig. 3 Temperature response of entrance face

where M_n, $n = 1,2,3, \ldots$ are the positive real eigenvalues given by the equation

$$[M_n^2 - (g^2/4)] \tan M_n = gM_n \qquad (6)$$

The steady-state temperature is given by the first term in Eq. (5).

Application

Eigenvalues M_n were calculated from Eq. (6) on an electronic digital computer for the first five roots and for $g = 0.01$, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 1.5, 2.0, 3.0, and 5.0.‡

For ease in graphical presentation, Eq. (5) was normalized by dividing both sides of the equation by the steady-state solution. Figures 1–3 present the normalized form of Eq. (5) for the coolant exit, center, and coolant entrance face, respectively. The ordinate then may be thought of as a fraction of steady-state temperature. Note that steady-state conditions are reached in shorter time as the mass flow parameter g increases. This is as expected, since the case $g = 0$ has no steady-state solution.

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$$u = \frac{1}{g}\exp[-g(1-\xi)] - \frac{-4}{g}\sum_{n=1}^{\infty}\frac{M_n^2 \sin M_n \exp\{-[(g^2/4) + M_n^2]F_0 - (g/2)(1-\xi)\}}{[(g^2/4) + M_n^2]\{2M_n - [(g^2/4) + M_n^2]\sin 2M_n/[(g^2/4) - M_n^2]\}}\left(\frac{g}{2M_n}\sin M_n\xi + \cos M_n\xi\right) \qquad (5)$$

† Derivation is available on request from the author.

‡ These data are available on request from the author.

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Large Thermal Deflection of a Cantilever Beam

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Nonlinear bending of a straight cantilever beam subjected to a temperature distribution that is linear through the thickness and proportional to the cosine of the angle of deformation is analyzed. A closed-form solution is obtained, and numerical results are discussed.

Nomenclature

A = undeformed configuration
B = deformed configuration
h = half-depth of beam
k = coefficient of thermal expansion
L = length of beam
P = arbitrary point on deformed middle surface
s = arc length measured along middle surface
T = temperature above a fixed datum
T_0 = reference temperature
x,y = coordinates of point P
z = perpendicular distance from middle surface
ϵ = longitudinal fiber strain
θ = angle between middle surface normal and y axis

INTEREST in problems concerning theory of the elastica has been widespread among applied mathematicians and engineers from the time of the classical investigations of Bernoulli[1] and Euler[2] to the contemporary studies of Love[3] and others.[4–13] As pointed out by Mitchell,[4] problems involving nonlinear bending of beams are of two basic types; viz., given the free shape and loads, find the deflected shape, or, given the deflected shape and loads, find the free shape. General solutions of the former problem usually entail considerable mathematical difficulty, and only in special cases can a solution be found by other than numerical methods. However, the latter problem, which seldom occurs in practice, possesses solutions that are relatively easy to derive.

The large deflection of a straight horizontal cantilever beam subjected to a vertical point load has been analyzed by Barten[5] and Bisshopp and Drucker.[6] Values of the free-end displacements of a circular-arc cantilever under vertical and horizontal point loads have been given by Conway.[7] Nonlinear bending of a straight horizontal cantilever under uniformly distributed load has been treated by Hummel and Morton,[8] Bickley,[9] Rohde,[10] and Truesdell.[11] Recently, Mitchell[4] presented a unified and more general treatment of several cases considered in Refs. 5–11.

The present note deals with nonlinear bending of a straight cantilever beam subjected to a temperature distribution that is linear through the thickness and proportional to the cosine of the angle of deformation. A closed-form solution is obtained, and numerical results are presented in nondimensional form.

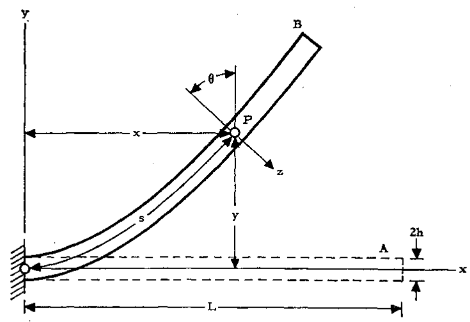

Fig. 1 Geometry of deformation

Basic Equations

Let the cantilever beam shown in Fig. 1 have length L and depth $2h$. Assume that the beam is bent in a principal plane from the undeformed and unstressed configuration A to the deformed configuration B in such a way that the middle surface does not stretch or twist. Cartesian coordinates of an arbitrary point P on the deformed middle surface are denoted by (x,y), s denotes arc length measured along the middle surface, z is the perpendicular distance from the middle surface, and θ represents the angle between a middle surface normal and the y axis.

Consequently, the engineering strain ϵ of a longitudinal fiber may be written

$$\epsilon = z(d\theta/ds) \tag{1}$$

Let the beam be exposed to a thermal environment such that its temperature T above some fixed datum is given by the relation

$$T = T_0(z/h)\cos\theta \tag{2}$$

where T_0 = const is a reference temperature on the lower surface of the clamped end of the beam. The end $s = L$ is not restrained, and consequently, within the framework of the present theory, the beam will remain stress-free.[14] Thus Hooke's law and Eq. (2) yield

$$\epsilon = kT_0(z/h)\cos\theta \tag{3}$$

where k is the coefficient of thermal expansion.

Necessary differential equations for the determination of (x,y,θ) follow from geometry of the elastic line (i.e., middle surface) and a comparison of Eqs. (1) and (3), viz.,

$$\begin{aligned} dx/ds &= \cos\theta \\ dy/ds &= \sin\theta \\ d\theta/ds &= (kT_0/h)\cos\theta \end{aligned} \tag{4}$$

Results

A quadrature of Eqs. (4) subject to the boundary conditions

$$x(0) = y(0) = \theta(0) = 0 \tag{5}$$

gives

$$\begin{aligned} \frac{x}{L} &= \frac{2}{kT_0L/h}\left[\text{arc tan}\left(\exp\frac{kT_0s}{h}\right) - \frac{\pi}{4}\right] \\ \frac{y}{L} &= \frac{1}{kT_0L/h}\ln\left(\cosh\frac{kT_0s}{h}\right) \\ \theta &= \text{arc sin}(\tanh kT_0s/h) \end{aligned} \tag{6}$$

Dimensionless plots of representative elastic curves based on the first two of Eqs. (6) are constructed as shown in Fig. 2.

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